1. We know that $\{x_k\}$ is Cauchy, therefore it is bounded. Assume that $\{x_k\}$ has a convergent subsequence $\{x_{k_n}\}$. Pick $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for any $n \geq N$, $d(x_{k_n}, x) < \varepsilon/2$. Let $N = \max \{N_1, N_2, N_3\}$. For any $n \geq N$, $d(x_n, x) < d(x_{k_n}, x) + d(x_{k_n}, x) = \varepsilon/2 + \varepsilon/2 = \varepsilon$. Therefore $x_n$ also converges and it converges to $x$.

2. We know that $\limsup_{n \to \infty} \left( \sup_{k \geq n} x_k \right)$ is monotonically non-increasing and $\lim_{n \to \infty} \sup_{k \geq n} x_k$ is monotonically non-decreasing. $(\sup_{n \geq 1} x_n)$ and $(\inf_{n \geq 1} x_n)$ are sequences and we are given that their limits are both finite. For $n \geq 1$, for all $n \geq 1$, and $\inf_{n \geq 1} x_n < x_n < \sup_{n \geq 1} x_n$, then $x_n$ is Cauchy.

3. Fix $\varepsilon > 0$. Since $\{x_k\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. Since $\{a_k\}$ is a finite set of points, it is bounded.

Since both $\sup_{n \geq 1} x_n$ and $\inf_{n \geq 1} x_n$ are bounded, this means that the original sequence $\{x_n\}$ is also bounded.

4. $\lim_{n \to \infty} \sup_{k \geq n} (a_k + b_k) = \lim_{n \to \infty} \left( \sup_{k \geq n} a_k + \sup_{k \geq n} b_k \right) \leq \lim_{n \to \infty} \sup_{k \geq n} a_k + \lim_{n \to \infty} \sup_{k \geq n} b_k$.

5. $\lim_{k \to \infty} \sup_{n \geq 1} (a_n + b_n)$
we know \( \lim \sup (\{a_n\}) = \lim \inf a_n \quad : \quad \lim \sup (- (a_n + b_n)) \geq - (\lim \sup a_n) + \lim \sup (b_n) \quad \Rightarrow \quad \lim \inf (a_n + b_n) \leq \lim \sup a_n + \lim \inf b_n \)