Problem 6.1

Proof by Induction. (i) Obviously the claim holds for $1 \times 1$ triangular matrices.

(ii) Assume that it holds for $k \times k$ triangular matrices. Let $A$ be a $(k+1) \times (k+1)$ triangular matrix and $a_{ij}$ be its $(i, j)$-th entry. We use $A_{(i,j)}$ to denote the submatrix of $A$ with the $i$-th row and the $j$-th column removed. Then we have

$$|A| = a_{11}|A_{(1,1)}| - a_{12}|A_{(1,2)}| + \cdots + (-1)^{k+1}a_{1k}|A_{(1,k)}|$$

Note that $A_{(1,1)}, \ldots, A_{(1,k)}$ are $k \times k$ triangular matrices and by our assumption, only

$$|A_{(1,1)}| = \prod_{1 \leq i \leq k} a_{ii}$$

and all other submatrices have determinant of zero. Therefore

$$|A| = \prod_{1 \leq i \leq k} a_{ii}$$

Problem 6.2

By the definition of eigenvectors, we have

$$As_j = \lambda_j s_j$$

Then

$$AS = A \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix} = \begin{bmatrix} \lambda_1 s_1 & \lambda_2 s_2 & \cdots & \lambda_n s_n \end{bmatrix} = \Lambda S$$

Since $\{s_1, \ldots, s_n\}$ are linearly independent, rank$(S) = n$ and therefore $S$ is invertible. We then have

$$S^{-1}AS = \Lambda$$